

# Conformal Brane World and Cosmological Constant

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## Abstract

We consider a recently proposed setup where a codimension one brane is embedded in the background of a smooth domain wall interpolating between AdS and Minkowski minima. Since the volume of the transverse dimension is infinite, bulk supersymmetry is intact even if brane supersymmetry is completely broken. On the other hand, in this setup unbroken bulk supersymmetry is incompatible with non-zero brane cosmological constant, so the former appears to protect the latter. In this paper we point out that, to have a consistent coupling between matter localized on the brane and bulk gravity, in this setup generically it appears to be necessary that the brane world-volume theory be conformal. Thus, unbroken bulk supersymmetry appears to actually protect not only the cosmological constant but also conformal invariance on the brane.

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## I. INTRODUCTION

In the Brane World scenario the Standard Model gauge and matter fields are assumed to be localized on branes (or an intersection thereof), while gravity lives in a larger dimensional bulk of space-time [1–12]. The volume of dimensions transverse to the branes is automatically finite if these dimensions are compact. On the other hand, the volume of the transverse dimensions can be finite even if the latter are non-compact. In particular, this can be achieved by using [13] warped compactifications [14] which localize gravity on the brane. A concrete realization of this idea was given in [15].

Recently it was pointed out in [16,17] that, in theories where extra dimensions transverse to a brane have infinite volume [18–22], the cosmological constant on the brane might be under control even if brane supersymmetry is completely broken. The key point here is that even if supersymmetry breaking on the brane does take place, it will not be transmitted to the bulk as the volume of the extra dimensions is infinite [16,17]. Thus, at least in principle, one should be able to control some of the properties of the bulk with the unbroken bulk supersymmetry. One then can wonder whether bulk supersymmetry could also control the brane cosmological constant [16,17].

Controlling the brane cosmological constant with bulk supersymmetry, however, appears to be non-trivial. Thus, in [23] it was pointed out that in the Dvali-Gabadadze-Porrati model [22], where a 3-brane is embedded in the 5-dimensional Minkowski space, unbroken bulk supersymmetry is perfectly compatible with non-vanishing, in particular, positive brane cosmological constant. There is a simple reason for this. The bulk curvature in this model is constant, in particular, it vanishes. The Minkowski space can be foliated by codimension one surfaces of vanishing or positive constant curvature. Both of these types of foliations are compatible with bulk supersymmetry, that is, with the existence of Killing spinors in the bulk - the latter is a (local) property of the corresponding space-time, and is independent of the foliation. Note that this also applies to another space of constant curvature which admits Killing spinors, namely, the AdS space. In this case we have foliations with vanishing, positive and negative constant curvature, all of which are perfectly consistent with bulk supersymmetry.

It is then natural to consider examples where the bulk is a space with non-constant curvature, which, nonetheless, admits Killing spinors. It is clear that (in the codimension one case) such a space would have half of the supersymmetries compared with the constant curvature cases. One way to parametrize such a space is to consider a bulk theory where gravity is coupled to a scalar field  $\phi$  with a non-trivial scalar potential  $V(\phi)$  (that is,  $V(\phi)$  is not a constant). For an appropriately chosen scalar potential there exist BPS domain wall solutions which preserve half of the original supersymmetries (that is, half of the supersymmetries corresponding to Minkowski or AdS minima of the scalar potential). The corresponding foliation is then necessarily flat. The reason why this is so different from the cases where the bulk curvature is constant is that in the latter cases one only needs to ensure vanishing of the gravitino variation, while in the presence of a non-trivial bulk potential preserving bulk supersymmetry also requires vanishing of the variation of the superpartner of  $\phi$ .

This is precisely the proposal of [24]. More concretely, let a codimension one brane (which, for simplicity, is taken to be  $\delta$ -function-like) be embedded in the BPS domain wall

background of the aforementioned type. To have an infinite volume extra dimension, the scalar potential is assumed to have one Minkowski minimum and one AdS minimum, and the domain wall interpolates between these minima. Note that such a domain wall (unlike a domain wall interpolating between two Minkowski minima [17,19]) does not have to violate the weak energy condition. In particular, if the domain wall is smooth, which is the case if the brane cosmological constant equals the brane tension, then the weak energy condition is not violated [24]. Now, since the volume of the extra dimension is infinite, supersymmetry breaking on the brane does not result in bulk supersymmetry breaking. However, the latter is not compatible with non-zero brane cosmological constant. Thus, even if supersymmetry breaking on the brane does occur, the brane cosmological constant appears to remain zero.

The purpose of this paper, which is essentially a follow-up of [24], is to point out that the theory living on the brane in the setup of [24] generically appears to be conformal. One way to see this is to consider small fluctuations around the background in the presence of matter localized on the brane. Then the system of equations for the small fluctuations of the scalar field and the metric is essentially overconstrained, and has consistent solutions if and only if the energy-momentum tensor of the matter localized on the brane is traceless and the coupling of the scalar field to the brane matter is vanishing. This appears to be due to the fact that a tensionless brane by itself does not explicitly break diffeomorphism invariance, so that the latter is broken spontaneously by the domain wall solution. Spontaneous breaking of translational invariance results in the gravitational Higgs mechanism discussed in [25], in the process of which the trace part  $h^\mu_\mu$  of the graviton roughly becomes a pure gauge. This then implies that in such a setup a consistent coupling of bulk gravity to the brane matter is possible only if the latter is conformal.

Thus, the approach of [24] might provide a realization of a setup where a conformal theory on the brane is gravitationally coupled to a non-conformal theory in the bulk. The conformal property of the brane world-volume theory is then preserved due to bulk supersymmetry, which is unbroken even if the brane theory is non-supersymmetric as the volume of the extra dimension is infinite. In particular, if gravitational interactions localized on the brane are generated via, say, loop diagrams, then the corresponding brane world gravity is expected to be conformal as well.

The remainder of this paper is organized as follows. In section II we briefly review the setup of [24]. In section III we study small fluctuations around the solution in the presence of brane matter sources, and discuss the requirement that the brane matter be conformal. Section IV contains concluding remarks.

## II. THE SETUP

In this section we review the setup of [24]. Thus, consider the model with the following action (more precisely, here we give the part of the action relevant for the subsequent discussions):

$$S = \widehat{M}_P^{D-3} \int_\Sigma d^{D-1}x \sqrt{-\widehat{G}} [\widehat{R} - \widehat{\Lambda}] + M_P^{D-2} \int d^Dx \sqrt{-G} \left[ R - \frac{4}{D-2} (\nabla\phi)^2 - V(\phi) \right]. \quad (1)$$

For calculational convenience we will keep the number of space-time dimensions  $D$  unspecified. In (1)  $\widehat{M}_P$  is (up to a normalization factor - see below) the  $(D-1)$ -dimensional

(reduced) Planck scale, while  $M_P$  is the  $D$ -dimensional one. The  $(D - 1)$ -dimensional hypersurface  $\Sigma$ , which we will refer to as the brane, is the  $y = y_0$  slice of the  $D$ -dimensional space-time, where  $y \equiv x^D$ , and  $y_0$  is a constant. Next,

$$\hat{G}_{\mu\nu} \equiv \delta_\mu^M \delta_\nu^N G_{MN} \Big|_{y=y_0} , \quad (2)$$

where the capital Latin indices  $M, N, \dots = 1, \dots, D$ , while the Greek indices  $\mu, \nu, \dots = 1, \dots, (D - 1)$ . The quantity  $\hat{\Lambda}$  is the brane tension. More precisely, there might be various (massless and/or massive) fields (such as scalars, fermions, gauge vector bosons, *etc.*), which we will collectively denote via  $\Phi^i$ , localized on the brane. Then  $\hat{\Lambda} = \hat{\Lambda}(\Phi^i, \nabla_\mu \Phi^i, \dots)$  generally depends on the vacuum expectation values of these fields as well as their derivatives. In the following we will assume that the expectation values of the  $\Phi^i$  fields are dynamically determined, independent of the coordinates  $x^\mu$ , and consistent with  $(D - 1)$ -dimensional general covariance. The quantity  $\hat{\Lambda}$  is then a constant which we identify as the brane tension. The bulk fields are given by the metric  $G_{MN}$ , a single real scalar field  $\phi$ , as well as other fields (whose expectation values we assume to be vanishing) which would appear in a concrete supergravity model (for the standard values of  $D$ ).

Let us briefly comment on the  $\sqrt{-\hat{G}}\hat{R}$  term in the brane world-volume action. Typically such a term is not included in discussions of various brane world scenarios (albeit usually the  $-\sqrt{-\hat{G}}\hat{\Lambda}$  term is). However, as was pointed out in [22], even if such a term is absent at the tree level, as long as the brane world-volume theory is not conformal, it will typically be generated by quantum loops of other fields localized on the brane (albeit not necessarily with the desired sign, which, nonetheless, appears to be as generic as the opposite one). This is an important observation, which allows to reproduce the  $(D - 1)$ -dimensional Newton's law on, say, a non-conformal brane embedded in  $D$ -dimensional Minkowski space-time [22]. However, as we will see in the following, in the above setup the brane world-volume theory is actually conformal, and  $\hat{M}_P = 0$ .

To proceed further, we will need equations of motion following from the action (1). Here we are interested in studying possible solutions to these equations which are consistent with  $(D - 1)$ -dimensional general covariance. That is, we will be looking for solutions with the warped metric of the following form:

$$ds_D^2 = \exp(2A) \tilde{g}_{\mu\nu} dx^\mu dx^\nu + dy^2 , \quad (3)$$

where the warp factor  $A$  and the scalar field  $\phi$ , which are functions of  $y$ , are independent of the coordinates  $x^\mu$ , and the  $(D - 1)$ -dimensional metric  $\tilde{g}_{\mu\nu}$  is independent of  $y$ . With this ansatz, we have the following equations of motion for  $\phi$  and  $A$ :

$$\frac{8}{D-2} [\phi'' + (D-1)A'\phi'] - V_\phi - Lf_\phi \delta(y - y_0) = 0 , \quad (4)$$

$$(D-1)(D-2)(A')^2 - \frac{4}{D-2}(\phi')^2 + V - \frac{D-1}{D-3}\tilde{\Lambda} \exp(-2A) = 0 , \quad (5)$$

$$(D-2)A'' + \frac{4}{D-2}(\phi')^2 + \frac{1}{D-3}\tilde{\Lambda} \exp(-2A) + \frac{1}{2}Lf\delta(y - y_0) = 0 . \quad (6)$$

Here

$$f \equiv \widehat{\Lambda} - \widetilde{\Lambda} \exp[-2A(y_0)] \quad (7)$$

is the effective brane tension. The scale  $L$ , defined as

$$L \equiv \widehat{M}_P^{D-3}/M_P^{D-2} , \quad (8)$$

plays the role of the crossover distance scale below which gravity is effectively  $(D-1)$ -dimensional, while above this scale it becomes  $D$ -dimensional. Next,  $\widetilde{\Lambda}$  is independent of  $x^\mu$  and  $y$ . In fact, it is nothing but the cosmological constant of the  $(D-1)$ -dimensional manifold, which is therefore an Einstein manifold, corresponding to the hypersurface  $\Sigma$ . Our normalization of  $\widetilde{\Lambda}$  is such that the  $(D-1)$ -dimensional metric  $\widetilde{g}_{\mu\nu}$  satisfies Einstein's equations:

$$\widetilde{R}_{\mu\nu} - \frac{1}{2}\widetilde{g}_{\mu\nu}\widetilde{R} = -\frac{1}{2}\widetilde{g}_{\mu\nu}\widetilde{\Lambda} . \quad (9)$$

Here we note that in the bulk (that is, for  $y \neq y_0$ ) one of the second order equations is automatically satisfied once the first order equation (5) as well as the other second order equation are satisfied. As usual, this is a consequence of Bianchi identities.

Note that by rescaling the coordinates  $x^\mu$  on the brane we can always set  $\exp[A(y_0)] = 1$ . Then the  $(D-1)$ -dimensional Planck scale is simply  $\widehat{M}_P$ . Let  $\phi_0 \equiv \phi(y_0)$ . Note that the above system of equations has smooth solutions for

$$f(\phi_0) = f_\phi(\phi_0) = 0 , \quad (10)$$

that is, if the brane cosmological constant and the brane tension are equal

$$\widetilde{\Lambda} = \widehat{\Lambda} , \quad (11)$$

and there is no  $\phi$  tadpole due to the brane. In particular, in these solutions  $\phi$  and  $A$  as well as their derivatives  $\phi'$  and  $A'$  are smooth.

Let us now discuss possible solutions of the above system of equations (4), (5) and (6) for  $f(\phi_0) = f_\phi(\phi_0) = 0$ . To obtain an infinite volume solution, let us assume that the scalar potential has one AdS minimum located at  $\phi = \phi_-$  and one Minkowski minimum located at  $\phi = \phi_+$  (without loss of generality we will assume that  $\phi_+ > \phi_-$ ). Moreover, let us assume that there are no other extrema except for a dS maximum located at  $\phi = \phi_*$ , where  $\phi_- < \phi_* < \phi_+$ , such that  $V(\phi_*) \gg V(\phi_+) - V(\phi_-) = |V(\phi_-)|$ . This latter condition is necessary to sufficiently suppress the probability for nucleation of AdS bubbles in the Minkowski vacuum, which could otherwise destabilize the background [26]. Then we can have smooth domain walls interpolating between the two vacua. In fact, for  $\widetilde{\Lambda} = 0$  we have  $\phi(y) \rightarrow \phi_\pm$  as  $y \rightarrow \pm\infty$ . On the other hand, for  $\widetilde{\Lambda} > 0$  we have  $\phi(y) \rightarrow \phi_+$  as  $y \rightarrow +\infty$ , while  $\phi(y) \rightarrow \phi_-$  as  $y \rightarrow y_-$ , where  $y_- < y_0$  is finite (here for definiteness we have assumed that the domain wall approaches the Minkowski vacuum as  $y \rightarrow +\infty$ ). As to the warp factor  $A$ , it goes to  $-\infty$  as  $\phi \rightarrow \phi_-$  (if  $\widetilde{\Lambda} = 0$ , then  $A$  goes to  $-\infty$  linearly with  $|y|$ , while if  $\widetilde{\Lambda} > 0$ , then  $A \sim \ln(y - y_-)$  as  $y \rightarrow y_-$ ). On the other hand, if  $\widetilde{\Lambda} = 0$ , then  $A$  goes to a constant as  $y \rightarrow +\infty$ , while if  $\widetilde{\Lambda} > 0$ , then  $A$  grows logarithmically with  $y$ . In both cases the volume of the extra dimension is infinite as the integral

$$\int dy \exp[(D-1)A] \quad (12)$$

diverges. Moreover, there are no quadratically normalizable bulk graviton modes. Rather, for  $\tilde{\Lambda} = 0$  we have a continuum of plane-wave normalizable bulk modes (with mass squared  $m^2 \geq 0$ ), while for  $\tilde{\Lambda} > 0$  we have a mass gap in the bulk graviton spectrum, and the plane-wave normalizable modes are those with  $m^2 > m_1^2$ , where  $m_1^2 \sim \tilde{\Lambda}$  [23]. Thus, without any additional assumptions consistent solutions with vanishing as well as positive brane cosmological constant exist for such potentials.

However, as was pointed out in [24], as long as the scalar potential  $V(\phi)$  is non-trivial, bulk supersymmetry is incompatible with non-zero brane cosmological constant. Indeed, this immediately follows from the bulk Killing spinor equations (following from the requirement that variations of the superpartner  $\lambda$  of  $\phi$  and the gravitino  $\psi_M$  vanish under the corresponding supersymmetry transformations), which in such backgrounds reduce to:

$$\phi' = \alpha W_\phi , \quad (13)$$

$$A' = \beta W , \quad (14)$$

where  $W$  is the superpotential,

$$\alpha \equiv \eta \frac{\sqrt{D-2}}{2} , \quad \beta \equiv -\eta \frac{2}{(D-2)^{3/2}} , \quad (15)$$

and  $\eta = \pm 1$ .

Note that the system of equations (13) and (14) is compatible with the system of equations (4), (5) and (6) if and only if  $\tilde{\Lambda} = 0$ , and the scalar potential is given by

$$V = W_\phi^2 - \gamma^2 W^2 , \quad (16)$$

where

$$\gamma \equiv \frac{2\sqrt{D-1}}{D-2} . \quad (17)$$

Thus, bulk supersymmetry (note that the domain wall solution preserves 1/2 of the supersymmetries corresponding to the minima of  $V$ ) is preserved if and only if the brane cosmological constant vanishes. We therefore conclude that even if brane supersymmetry is broken, bulk supersymmetry, which remains unbroken as the volume of the transverse dimension is infinite, ensures that the brane cosmological constant still vanishes in the model defined in (1).

Before we end this section, for illustrative purposes let us give an example of a domain wall of the aforementioned type. Let

$$W = \xi \left[ v^2 \phi - \frac{1}{3} \phi^3 - \frac{2}{3} v^3 \right] . \quad (18)$$

Note that at  $\phi_- = -v$  we have the AdS minimum, while at  $\phi_+ = +v$  we have the Minkowski minimum. To ensure that the condition  $|V(\phi_-)| \ll V(\phi_*)$  is satisfied, where  $\phi_*$  ( $\phi_- < \phi_* < \phi_+$ ) corresponds to the dS maximum, we must assume  $v \ll 1$ . The domain wall solution, which interpolates between the AdS and Minkowski vacua in this case is given by

$$\phi(y) = v \tanh(\alpha \xi v(y - y_1)) , \quad (19)$$

$$A(y) = \frac{2\beta}{3\alpha} v^2 \left[ \ln(\cosh(\alpha \xi v(y - y_1))) - \frac{1}{4 \cosh^2(\alpha \xi v(y - y_1))} \right] - \frac{2\beta}{3} \xi v^3 (y - y_1) + C , \quad (20)$$

where  $y_1$  and  $C$  are integration constants.

Finally, let us note that solutions with non-vanishing  $f(\phi_0)$  and  $f_\phi(\phi_0)$  do not interpolate between the AdS and Minkowski vacua. Thus, solutions with positive  $f(\phi_0)$  asymptotically approach the AdS minimum on both sides of the brane, while solutions with negative  $f(\phi_0)$  asymptotically approach the Minkowski minimum on both sides of the brane<sup>1</sup>.

### III. BRANE MATTER SOURCES

In this section we would like to study gravitational interactions between sources localized on the brane. To do this, let us start from the action (1), and study small fluctuations of the metric  $G_{MN}$  and the scalar field  $\phi$ , which we will denote via  $h_{MN}$  and  $\varphi$ , respectively, around the corresponding smooth domain wall solution (with vanishing brane cosmological constant) in the presence of brane matter sources.

In the following it will prove convenient to make the coordinate transformation  $y \rightarrow z$  so that the background metric takes the form:

$$ds_D^2 = \exp(2A) \left[ \eta_{\mu\nu} dx^\mu dx^\nu + dz^2 \right] . \quad (21)$$

That is,

$$dy = \exp(A) dz , \quad (22)$$

where we have chosen the overall sign so that  $z \rightarrow \pm\infty$  as  $y \rightarrow \pm\infty$ . Moreover, we can fix the integration constant upon solving (22) such that  $y = y_0$  is mapped to  $z = 0$ . So from now on we will use the coordinates  $x^M = (x^\mu, x^D) = (x^\mu, z)$ , and prime will denote derivative w.r.t.  $z$ . Moreover, the capital Latin indices  $M, N, \dots$  are lowered and raised with the flat  $D$ -dimensional Minkowski metric  $\eta_{MN}$  and its inverse, while the Greek indices  $\mu, \nu, \dots$  are lowered and raised with the flat  $(D - 1)$ -dimensional Minkowski metric  $\eta_{\mu\nu}$  and its inverse.

Also, instead of metric fluctuations  $h_{MN}$ , it will be convenient to work with  $\tilde{h}_{MN}$  defined via

$$h_{MN} = \exp(2A) \tilde{h}_{MN} . \quad (23)$$

It is not difficult to see that in terms of  $\tilde{h}_{MN}$  the  $D$ -dimensional diffeomorphisms

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<sup>1</sup>Note that  $f(\phi_0)$  is the effective brane tension. If  $f(\phi_0) < 0$ , then we have world-volume ghosts unless we assume that the brane is an “end-of-the-world” brane located at an orbifold fixed point. Thus, in solutions with  $f(\phi_0) < 0$  the geometry of the  $y$  dimension is that of  $\mathbf{R}/\mathbf{Z}_2$  (and not of  $\mathbf{R}$ ), with the orbifold fixed point identified with  $y_0$  (then the corresponding solution on the covering space has the  $\mathbf{Z}_2$  symmetry required for the orbifold interpretation), and the brane is stuck at the orbifold fixed point.

$$\delta h_{MN} = \nabla_M \xi_N + \nabla_N \xi_M \quad (24)$$

are given by the following gauge transformations:

$$\delta \tilde{h}_{MN} = \partial_M \tilde{\xi}_N + \partial_N \tilde{\xi}_M + 2A' \eta_{MN} \tilde{\xi}_S n^S. \quad (25)$$

Here for notational convenience we have introduced a unit vector  $n^M$  with the following components:  $n^\mu = 0$ ,  $n^D = 1$ .

### A. Equations of Motion

To proceed further, we need equations of motion for  $\tilde{h}_{MN}$  and  $\varphi$ . Let us assume that we have matter localized on the brane, and let the corresponding conserved energy-momentum tensor be  $T_{\mu\nu}$ :

$$\partial^\mu T_{\mu\nu} = 0. \quad (26)$$

The graviton field  $\tilde{h}_{\mu\nu}$  couples to  $T_{\mu\nu}$  via the following term in the action:

$$S_{\text{int}} = \int_\Sigma d^{D-1}x \left[ \frac{1}{2} T_{\mu\nu} \tilde{h}^{\mu\nu} + \frac{8}{D-2} \Theta \varphi \right], \quad (27)$$

where we have also included the corresponding coupling of  $\varphi$  to the brane matter. Next, starting from the action  $S + S_{\text{int}}$  we obtain the following linearized equations of motion for  $\tilde{h}_{MN}$  and  $\varphi$ :

$$\begin{aligned} & \left\{ \partial_S \partial^S \tilde{h}_{MN} + \partial_M \partial_N \tilde{h} - \partial_M \partial^S \tilde{h}_{SN} - \partial_N \partial^S \tilde{h}_{SM} - \eta_{MN} [\partial_S \partial^S \tilde{h} - \partial^S \partial^R \tilde{h}_{SR}] \right\} + \\ & (D-2)A' \left\{ [\partial_S \tilde{h}_{MN} - \partial_M \tilde{h}_{NS} - \partial_N \tilde{h}_{MS}] n^S + \eta_{MN} [2\partial^R \tilde{h}_{RS} - \partial_S \tilde{h}] n^S \right\} - \\ & \eta_{MN} \tilde{h}_{SR} n^S n^R V \exp(2A) = \\ & \frac{8}{D-2} \phi' [\eta_{MN} \partial_S \varphi n^S - \partial_M \varphi n_N - \partial_N \varphi n_M] + \eta_{MN} \varphi V_\phi \exp(2A) - M_P^{2-D} \tilde{T}_{MN} \delta(z), \end{aligned} \quad (28)$$

$$\begin{aligned} & \partial_S \partial^S \varphi + (D-2)A' \partial_S \varphi n^S - \frac{D-2}{8} \varphi V_{\phi\phi} \exp(2A) - \frac{1}{2} \phi' [2\partial^R \tilde{h}_{RS} - \partial_S \tilde{h}] n^S - \\ & \frac{D-2}{8} \tilde{h}_{SR} n^S n^R V_\phi \exp(2A) = -M_P^{2-D} \tilde{\Theta} \delta(z), \end{aligned} \quad (29)$$

where  $\tilde{h} \equiv \tilde{h}_M^M$ ,  $\tilde{T}_{MN} \equiv T_{MN} + T_{MN}^{\text{brane}}$ ,  $\tilde{\Theta} \equiv \Theta + \Theta^{\text{brane}}$ . Here  $T_{MN}^{\text{brane}}$  and  $\Theta^{\text{brane}}$  are the corresponding brane contributions (which are linear in  $\tilde{h}_{MN}$  and  $\varphi$ ) coming from the first term in (1)<sup>2</sup>. Note that the only non-vanishing components of  $\tilde{T}_{MN}$  are  $\tilde{T}_{\mu\nu}$ , and we have  $\partial^\mu \tilde{T}_{\mu\nu} = 0$ .

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<sup>2</sup>If the brane world-volume theory is not conformal, then we can *a priori* expect that a kinetic term for the  $\phi$  field will also be generated on the brane (just as it happens for the graviton). Then  $\Theta^{\text{brane}}$  also contains a term proportional to  $\partial^\mu \partial_\mu \varphi$  along with the term proportional to  $f_{\phi\phi}(\phi_0)\varphi$ . However, at the end of the day we will find that the brane world-volume theory is conformal, so these terms are not generated by quantum effects.



The above equations of motion drastically simplify if we perform a gauge transformation (25) with

$$\tilde{\xi}_M = n_M(\varphi/\phi') . \quad (30)$$

The new equations of motion then read

$$\begin{aligned} & \left\{ \partial_S \partial^S \tilde{h}_{MN} + \partial_M \partial_N \tilde{h} - \partial_M \partial^S \tilde{h}_{SN} - \partial_N \partial^S \tilde{h}_{SM} - \eta_{MN} [\partial_S \partial^S \tilde{h} - \partial^S \partial^R \tilde{h}_{SR}] \right\} + \\ & (D-2)A' \left\{ [\partial_S \tilde{h}_{MN} - \partial_M \tilde{h}_{NS} - \partial_N \tilde{h}_{MS}] n^S + \eta_{MN} [2\partial^R \tilde{h}_{RS} - \partial_S \tilde{h}] n^S \right\} - \\ & \eta_{MN} \tilde{h}_{SR} n^S n^R V \exp(2A) = -M_P^{2-D} \tilde{T}_{MN} \delta(z) , \end{aligned} \quad (31)$$

$$-\frac{1}{2}\phi' [2\partial^R \tilde{h}_{RS} - \partial_S \tilde{h}] n^S - \frac{D-2}{8} \tilde{h}_{SR} n^S n^R V_\phi \exp(2A) = -M_P^{2-D} \tilde{\Theta} \delta(z) . \quad (32)$$

Note that these equations of motion no longer contain  $\varphi$ . This has a simple physical interpretation [25]. The domain wall background spontaneously breaks translational invariance in the  $z$  direction. Since this invariance is a gauge symmetry, the corresponding Goldstone mode, which is given by configurations where  $\omega \equiv \varphi/\phi'$  is independent of  $z$  [25]<sup>3</sup>, must be eaten in the corresponding Higgs mechanism. The field which eats the Goldstone mode is nothing but the graviphoton  $h_{\mu D}$  arising in the decomposition of the  $D$ -dimensional metric fluctuations in terms of  $(D-1)$ -dimensional fields [25]. Note, however, that with the above gauge fixing not only the Goldstone zero mode but all  $\varphi$  modes have been eliminated. There is, however, a price we have to pay for this simplification. In particular, the residual gauge invariance which preserves the equations of motion (31) and (32) is given by

$$\delta \tilde{h}_{MN} = \partial_M \tilde{\xi}_N + \partial_N \tilde{\xi}_M , \quad \tilde{\xi}_S n^S = 0 . \quad (33)$$

Note that here  $\tilde{\xi}_M$  need not be independent of  $z$ . Under these residual gauge transformations the fields  $\tilde{h}_{\mu\nu}, A_\mu, \rho$ , where  $A_\mu \equiv \tilde{h}_{\mu D}$  and  $\rho \equiv \tilde{h}_{DD}$ , transform as follows

$$\delta \tilde{h}_{\mu\nu} = \partial_\mu \tilde{\xi}_\nu + \partial_\nu \tilde{\xi}_\mu , \quad \delta A_\mu = \tilde{\xi}'_\mu , \quad \delta \rho = 0 . \quad (34)$$

This implies that we cannot gauge  $\rho$  away. We can, however, gauge  $A_\mu$  away. Thus, in the following we will use the gauge where  $A_\mu = 0$ . Note that after this gauge fixing the residual gauge transformations are given by

$$\delta \tilde{h}_{\mu\nu} = \partial_\mu \tilde{\xi}_\nu + \partial_\nu \tilde{\xi}_\mu , \quad \delta \rho = 0 , \quad \tilde{\xi}'_\mu = 0 . \quad (35)$$

We now have the following equations of motion:

$$\begin{aligned} & \{ \partial_\sigma \partial^\sigma H_{\mu\nu} + \partial_\mu \partial_\nu H - \partial_\mu \partial^\sigma H_{\sigma\nu} - \partial_\nu \partial^\sigma H_{\sigma\mu} - \eta_{\mu\nu} [\partial_\sigma \partial^\sigma H - \partial^\sigma \partial^\rho H_{\sigma\rho}] \} + \\ & \{ H''_{\mu\nu} - \eta_{\mu\nu} H'' + (D-2)A' [H'_{\mu\nu} - \eta_{\mu\nu} H'] \} + \\ & \{ \partial_\mu \partial_\nu \rho - \eta_{\mu\nu} \partial_\sigma \partial^\sigma \rho + \eta_{\mu\nu} [(D-2)A' \rho' - \rho V \exp(2A)] \} = -M_P^{2-D} \tilde{T}_{\mu\nu} \delta(z) , \end{aligned} \quad (36)$$

$$[\partial^\mu H_{\mu\nu} - \partial_\nu H]' + (D-2)A' \partial_\nu \rho = 0 , \quad (37)$$

$$-[\partial^\mu \partial^\nu H_{\mu\nu} - \partial^\mu \partial_\mu H] + (D-2)A' H' + \rho V \exp(2A) = 0 , \quad (38)$$

$$\phi' [H' - \rho'] - \frac{D-2}{4} \rho V_\phi \exp(2A) = -2M_P^{2-D} \tilde{\Theta} \delta(z) , \quad (39)$$

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<sup>3</sup>To see this, note that the translational Goldstone mode corresponds to fluctuations around the solution given by  $\phi(z + \omega(x^\mu)) = \phi(z) + \phi'(z)\omega(x^\mu) + \mathcal{O}(\omega^2)$ .

where  $H_{\mu\nu} \equiv \tilde{h}_{\mu\nu}$ , and  $H \equiv H^\mu_\mu$ .

Not all of the above equations are independent. First, differentiating (36) with  $\partial^\mu$  (and taking into account that  $\partial^\mu \tilde{T}_{\mu\nu} = 0$ ), we obtain an equation which is identically satisfied once we take into account (37) together with the equations of motion for  $A$  and  $\phi$ . Next, taking the trace in (36), we obtain an equation which together with (37), (38) and the equations of motion for  $A$  and  $\phi$  gives the following equation:

$$\frac{4}{D-2}(\phi')^2 [H' - \rho'] - \rho V_\phi \phi' \exp(2A) = M_P^{2-D} A' \tilde{T} \delta(z) , \quad (40)$$

where  $\tilde{T} \equiv \tilde{T}^\mu_\mu$ . This equation is compatible with (39) if and only if

$$\tilde{\Theta} = -\frac{D-2}{8} \frac{A'(0)}{\phi'(0)} \tilde{T} . \quad (41)$$

Thus, we already see that the coupling of the brane matter to the bulk scalar cannot be arbitrary but is determined in terms of the trace of the energy-momentum tensor. (Note that neither  $A'$  nor  $\phi'$  vanish anywhere in the backgrounds we consider here, including the location of the brane  $z = 0$ .)

Finally, let us discuss (39). It came from the equation of motion for  $\varphi$ , which was a second order equation. However, after we eliminated  $\varphi$  itself, this equation became a first order equation in terms of  $H$  and  $\rho$ . This then implies that the source term on the r.h.s. of (39) must vanish or else  $H - \rho$  will be discontinuous. Thus, we have arrived at the conclusion that consistency of the above equations implies that we must have

$$\tilde{T} = \tilde{\Theta} = 0 . \quad (42)$$

This, in particular, implies that the brane world-volume theory is generically expected to be conformal in this setup. Indeed, it is not difficult to see that (42) can be satisfied if  $T \equiv T^\mu_\mu$  as well as  $\Theta$  vanish. To ensure conformality of the matter localized on the brane then generically requires that the brane world-volume theory itself be conformal. On the other hand, if this is not the case then to satisfy (42)  $T$  and  $\Theta$  (in the best case where  $f_{\phi\phi}(\phi_0) = 0$ ) must be the same up to a non-vanishing constant<sup>4</sup>, which generically need not be the case.

Let us verify that, if (42) is satisfied, the above system of equations does have a consistent solution for  $H_{\mu\nu}$  and  $\rho$ . It is not difficult to show that such a solution indeed exists, and is given by (note that  $p^\mu \tilde{T}_{\mu\nu}(p) = 0$ )

$$\rho = 0 , \quad (43)$$

$$H_{\mu\nu}(p, z) = M_P^{2-D} \tilde{T}_{\mu\nu}(p) \Omega(p, z) , \quad (44)$$

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<sup>4</sup>To see this, note that, once we perform the gauge transformation (25) with the gauge parameter given in (30),  $\tilde{T}_{\mu\nu}$  contains a term proportional to  $(D-3) [\partial_\mu \partial_\nu \omega - \eta_{\mu\nu} \partial^\sigma \partial_\sigma \omega]$ . Assuming  $D \neq 2, 3$ , we then have that on the brane  $\partial^\sigma \partial_\sigma \omega$  is proportional to  $T$  with a non-vanishing coefficient. On the other hand, the condition  $\tilde{\Theta} = 0$  gives a second order differential equation on the brane for  $\omega$  with a source term proportional to  $\Theta$ . Hence the aforementioned conclusion about the relation between  $T$  and  $\Theta$ .

where we have performed a Fourier transform w.r.t. the coordinates  $x^\mu$  (and the corresponding momenta are  $p^\mu$ ). Also, let us Wick rotate to the Euclidean space (where the propagator is unique). The function  $\Omega(p, z)$  is a solution to the following equation ( $p^2 \equiv p^\mu p_\mu$ )

$$\Omega''(p, z) + (D - 2)A'\Omega'(p, z) - p^2\Omega(p, z) = -\delta(z) \quad (45)$$

subject to the boundary conditions (for  $p^2 > 0$ )

$$\Omega(p, z \rightarrow \pm\infty) = 0 . \quad (46)$$

The above solution describes a gravitational field of conformal matter localized on the brane.

## B. Additional Evidence

In this subsection we would like to give additional evidence that the condition  $\tilde{\Theta} = 0$  (from which it follows that  $\tilde{T} = 0$ ) is indeed necessary. To begin with, let us perform the aforementioned Fourier transform in (36), (37), (38) and (39), and Wick rotate to the Euclidean space. The equations of motion for the Fourier transformed quantities read:

$$\begin{aligned} & - \left\{ p^2 H_{\mu\nu} + p_\mu p_\nu H - p_\mu p^\sigma H_{\sigma\nu} - p_\nu p^\sigma H_{\sigma\mu} - \eta_{\mu\nu} [p^2 H - p^\sigma p^\rho H_{\sigma\rho}] \right\} + \\ & \left\{ H''_{\mu\nu} - \eta_{\mu\nu} H'' + (D - 2)A' [H'_{\mu\nu} - \eta_{\mu\nu} H'] \right\} + \\ & \left\{ -p_\mu p_\nu \rho + \eta_{\mu\nu} [(D - 2)A' \rho' + \rho (p^2 - V \exp(2A))] \right\} = -M_P^{2-D} \tilde{T}_{\mu\nu}(p) \delta(z) , \end{aligned} \quad (47)$$

$$[p^\mu H_{\mu\nu} - p_\nu H]' + (D - 2)A' p_\nu \rho = 0 , \quad (48)$$

$$[p^\mu p^\nu H_{\mu\nu} - p^2 H] + (D - 2)A' H' + \rho V \exp(2A) = 0 , \quad (49)$$

$$\frac{4}{D - 2} (\phi')^2 [H' - \rho'] - \frac{D - 2}{4} \rho V_\phi \phi' \exp(2A) = M_P^{2-D} A' \tilde{T}(p) \delta(z) , \quad (50)$$

where we have taken into account (41).

Let us assume that  $\tilde{T}(p) \neq 0$ . Then the most general tensor structure for the fields  $H_{\mu\nu}$  and  $\rho$  can be parametrized in terms of four functions  $a, b, c, d$  as follows:

$$\rho = M_P^{2-D} d \tilde{T}(p) , \quad (51)$$

$$H_{\mu\nu} = M_P^{2-D} \left\{ a \tilde{T}_{\mu\nu}(p) + [b \eta_{\mu\nu} + c p_\mu p_\nu] \tilde{T}(p) \right\} . \quad (52)$$

Plugging this back into the equations of motion, we obtain six equations for four unknowns  $a, b, c, d$ . However, as should be clear from the above discussion, two of them are identically satisfied once we take into account the other four (as well as the equations of motion for  $A$  and  $\phi$ ). After some straightforward computations we obtain the following system of four independent equations:

$$a'' + (D - 2)A'a' - p^2 a = -\delta(z) , \quad (53)$$

$$(D - 2)A'd = a' + (D - 2)b' , \quad (54)$$

$$A' [(D - 2)p^2 c' - a'] = p^2 [a + (D - 2)b] - \frac{4}{D - 2} (\phi')^2 d , \quad (55)$$

$$a + (D - 3)b - c'' - (D - 2)A'c' + d = 0 . \quad (56)$$

Note that from the first equation it follows that  $a(p, z) = \Omega(p, z)$ .

Let  $w \equiv a + (D - 2)b$ . Using the above equations for  $a, b, c, d$  after some straightforward computations we obtain the following second order equation for  $w$ :

$$w'' + \left\{ (D - 2)A' - [\ln(F)]' \right\} w' - p^2 w = -F\delta(z) , \quad (57)$$

where  $F(z)$  is the following function:

$$F \equiv \frac{(A')^2}{(A')^2 - A''} . \quad (58)$$

Solutions to the above equation for  $w$  have some peculiar properties. To expose them, we need to study the asymptotic behavior of the function  $[\ln(F)]'$ .

To begin with, it is not difficult to see that

$$F = -\frac{\beta}{\alpha} \frac{W^2}{(W_\phi)^2} , \quad (59)$$

where the superpotential  $W$  as well as constants  $\alpha, \beta$  were defined in section II. It then follows that

$$[\ln(F)]' = 2\alpha \exp(A) \frac{(W_\phi)^2 - WW_{\phi\phi}}{W} . \quad (60)$$

Let us compute this function in the example discussed at the end of section II. In that example the superpotential is given by (18). We then have

$$[\ln(F)]' = -2\alpha\xi v \exp(A) \frac{3 + 2\tilde{\phi} + \tilde{\phi}^2}{2 + \tilde{\phi}} , \quad (61)$$

where  $\tilde{\phi} \equiv \phi/v$ . For definiteness let us assume that  $\alpha\xi v > 0$ . Then at  $z \rightarrow +\infty$  the domain wall solution approaches the Minkowski vacuum where  $\tilde{\phi} = +1$ , while at  $z \rightarrow -\infty$  it approaches the AdS vacuum where  $\tilde{\phi} = -1$ . It then follows that  $[\ln(F)]'$  is always negative on the solution. Moreover, at  $z \rightarrow +\infty$  (where  $A$  goes to a constant and  $A'$  goes to zero)  $[\ln(F)]'$  goes to a constant, which we will denote by  $-2\zeta$ . Then for large positive  $z$   $w$  is well approximated by the solution to the following equation

$$w'' + 2\zeta w' - p^2 w = 0 \quad (62)$$

subject to the boundary condition  $w(z \rightarrow +\infty) = 0$ . This solution is given by

$$w(z) = \text{const.} \times \exp(-\lambda z) , \quad (63)$$

where (the other root of the corresponding quadratic equation is negative)

$$\lambda \equiv \zeta + \sqrt{\zeta^2 + p^2} . \quad (64)$$

Note that for  $p \rightarrow 0$   $\lambda$  does not vanish but approaches  $2\zeta$ . This implies that even at zero momentum there is a non-trivial solution to (57)<sup>5</sup>. In particular, it is given by  $w(z) = \tilde{w}(0)$  for  $z < 0$ ,  $w(z) = \tilde{w}(z)$  for  $z \geq 0$ , where  $\tilde{w}(z)$  is the solution of the equation

$$\tilde{w}'' + \left\{ (D-2)A' - [\ln(F)]' \right\} \tilde{w}' - p^2 \tilde{w} = 0 \quad (65)$$

subject to the boundary conditions  $\tilde{w}(z \rightarrow +\infty) = 0$ , and  $\tilde{w}'(0) = -F(0)$  (note that  $F(z)$  is always positive). The fact that such a solution always exists for  $v \ll 1$  can be seen from the fact that in this case  $(D-2)A' \ll -[\ln(F)]'$ .

The existence of a non-trivial solution at  $p^2 = 0$  indicates an inconsistency in the system. Note that if we have  $\tilde{T} = 0$  to begin with, then we do not have the same system of equations for  $a, b, c, d$  as above. In fact in this case there is no inconsistency, and we have a consistent solution discussed in the previous subsection.

#### IV. REMARKS

Thus, as we see, in the setup of [24], where the brane cosmological constant is protected by bulk supersymmetry, to have consistent couplings of the bulk scalar and gravity to matter localized on the brane, it appears to be necessary that the latter is conformal. This then implies that (generically) the brane world-volume theory should itself be conformal. The fact that the brane cosmological constant vanishes is then a trivial consequence of conformal invariance of the brane world-volume theory. However, what appears to be non-trivial is that unbroken bulk supersymmetry in this setup (where the volume of the extra dimension is infinite) actually protects conformality of the brane world-volume theory (which *a priori* need not even be supersymmetric).

In this context one might hope to use this setup as a possible realization of the conformal approach to phenomenology [27] (also see [28]). However, it is still unclear how one could possibly have the brane conformal invariance broken around TeV while having much larger Planck scale on the brane.

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<sup>5</sup>Here we should point out that this does not occur if we consider domain walls arising for runaway type of potentials discussed in [24]. Nonetheless, the fact that a non-trivial solution of (57) does exist in the above example if we assume  $\tilde{T} \neq 0$  might be considered as (at least indirect) evidence that  $\tilde{T} = 0$  condition is indeed necessary. At any rate, if this condition is not satisfied, as we have already pointed out, there is a discontinuity in  $H - \rho$ , which appears to be inconsistent.

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